Systems of Linear Equations (SLE)

In mathematics, a **system of linear equations** (or **linear system**) is a collection of one or more linear equations involving the same set of variables.

Computational algorithms for finding the solutions are an important part of numerical linear algebra, and play a prominent role in engineering, physics, chemistry, computer science, and economics.

Examples: $$\begin{cases} x + 2y + z = 6 \\ 5x + 3y - z = 7 \\ 2x - y + z = 2 \end{cases}$$

General form $$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

Where $x_1, x_2,.., x_n$ are the unknowns, $a_{11}, a_{12},\ldots, a_{mn}$ are the coefficients of the system, and $b_1, b_2,\ldots, b_m$ are the constant terms.

The system of linear equations is equivalent to a matrix equation of the form $\boldsymbol{Ax = b}$ where A is an $m \times n$ matrix, **x** is a column vector with n entries, and **b** is a column vector with m entries.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ and } b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Solution of **(SLE)**

Gaussian elimination

Gaussian elimination, also known as **row reduction**, is an algorithm in linear algebra for solving a system of linear equations.

Example:

Given equations are

$$6x + 3y + 2z = 6 \quad \text{...(i)}$$
$$6x + 4y + 3z = 0 \quad \text{...(ii)}$$
$$20x + 15y + 12z = 0 \quad \text{...(iii)}$$

To eliminate x, we are operating (ii) – (i), (iii) $-\frac{20}{6}$ (i), we obtain

$$6x + 3y + 2z = 6 \quad \text{...(i)}$$
$$y + z = -6 \quad \text{...(iv)}$$
$$5y + \frac{16}{3}z = -20 \quad \text{...(v)}$$

To eliminate y, we are operating (v) – 5 (iv), we get

$$6x + 3y + 2z = 6 \quad \text{...(i)}$$
$$y + z = -6 \quad \text{...(iv)}$$
$$\frac{1}{3}z = 10 \quad \text{...(vi)}$$

From equation (vi), we get $z = 30$

Putting $z = 30$ in equation (iv), we get

$$y + 30 = -6$$

or $y = -6 - 30$

or $= -36$

Putting $y = -36$, $z = 30$ in equation (i), we obtain

$$6x + 3(-36) + 2(30) = 6$$
$$6x - 108 + 60 = 6$$
$$6x = 6 + 108 - 60$$
$$6x = 54$$
$$x = 9$$

Hence $x = \mathbf{9}$, $y = \mathbf{-36}$, $z = \mathbf{30}$ **Ans.**

Example: Solve the SLE $\begin{cases} 6x_1 - 2x_2 + 2x_3 + 4x_4 = 16 \\ 12x_1 - 8x_2 + 6x_3 + 10x_4 = 26 \\ 3x_1 - 13x_2 + 9x_3 + 3x_4 = -19 \\ -6x_1 + 4x_2 + x_3 - 18x_4 = -34 \end{cases}$

Solution:

eliminate x_1 : $\begin{cases} 6x_1 - 2x_2 + 2x_3 + 4x_4 = 16 \text{ does not change} \\ -4x_2 + 2x_3 + 2x_4 = -6 \\ -12x_2 + 8x_3 + x_4 = -27 \\ 2x_2 + 3x_3 - 14x_4 = -18 \end{cases}$

Applying $R_2' = R_2 - 2R_1$, $R_3' = R_3 - \frac{1}{2}R_1$, $R_4' = R_4 + R_1$

eliminate x_2 : $\sim \begin{cases} 6x_1 - 2x_2 + 2x_3 + 4x_4 = 16 & \text{does not change} \\ -4x_2 + 2x_3 + 2x_4 = -6 & \text{does not change} \\ 2x_3 - 5x_4 = -9 \\ 4x_3 - 13x_4 = -21 \end{cases}$

Applying $R_3' = R_3 - 6R_2$, $R_4' = R_4 - \frac{1}{2}R_2$

eliminate x_3 : $\sim \begin{cases} 6x_1 - 2x_2 + 2x_3 + 4x_4 = 16 & \text{does not change} \\ -4x_2 + 2x_3 + 2x_4 = -6 & \text{does not change} \\ 2x_3 - 5x_4 = -9 & \text{does not change} \\ -3x_4 = -3 \end{cases}$

Applying $R_4' = R_4 - 2R_3$

Then the linear system become

$$\begin{cases} 6x_1 - 2x_2 + 2x_3 + 4x_4 = 16 \\ -4x_2 + 2x_3 + 2x_4 = -6 \\ 2x_3 - 5x_4 = -9 \\ -3x_4 = -3 \end{cases}$$

Then by back-substitution method we have $x_1 = 3,\ x_2 = 1, x_3 = -2, x_4 = 1$

Exercise : Gauss-Elimination	
(i) $10x + y + z = 12$ $x + 10y + z = 12$ $x + y + 10z = 12$	(ii) $x + 3y + 10z = 24$ $2x + 17y + 4z = 35$ $28x + 4y - z = 32$